Universidade Federal do Vale do São Francisco
Engenharia Civil
Métodos Matemáticos

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1$^{\text{o}}$ Semestre

Gabarito 2ª Prova **2025**
Data: Domingo, 01 de Junho **Turma E5**

Exercício 1 *Observe que*

$$
\begin{aligned}
\operatorname{sen}(\cos w) = 0 &\quad\Rightarrow\\
\frac{e^{\mathbf{i}\cos w} - e^{-\mathbf{i}\cos w}}{2\mathbf{i}} = 0 &\quad\Rightarrow\\
e^{\mathbf{i}\cos w} - \frac{1}{e^{\mathbf{i}\cos w}} = 0 &\quad\Rightarrow\\
e^{2\mathbf{i}\cos w} - 1 = 0 &\quad\Rightarrow\\
2\mathbf{i}\cos w = \ln 1 &\quad\Rightarrow\\
2\mathbf{i}\cos w = \ln|1| + \mathbf{i}\arg 1 &\quad\Rightarrow\\
2\mathbf{i}\cos w = 2k\pi\mathbf{i} &\quad\Rightarrow\\
\cos w = k\pi &\quad\Rightarrow\\
\frac{e^{w\mathbf{i}} + e^{-w\mathbf{i}}}{2} = k\pi &\quad\Rightarrow\\
e^{2w\mathbf{i}} + 1 = 2k\pi e^{w\mathbf{i}} &\quad\Rightarrow\\
e^{2w\mathbf{i}} - 2k\pi e^{w\mathbf{i}} + 1 = 0 &
\end{aligned}
$$

Considere

$$\xi = e^{w\mathbf{i}}$$

e perceba que a equação anterior torna-se

$$\xi^2 - 2k\pi\xi + 1 = 0$$

cujas soluções são

$$
\begin{aligned}
\xi_1 &= k\pi + \sqrt{k^2\pi^2 - 1}\\
\xi_2 &= k\pi - \sqrt{k^2\pi^2 - 1}
\end{aligned}
$$

Assim, para $\xi = \xi_1$, tem-se

$$
\begin{aligned}
\xi_1 &= e^{w\mathbf{i}} \Rightarrow\\
w\mathbf{i} &= \ln \xi_1 \Rightarrow\\
w &= -\mathbf{i}\ln\left(k\pi + \sqrt{k^2\pi^2 - 1}\right)\\
&= -\mathbf{i}\left[\ln\left|k\pi + \sqrt{k^2\pi^2 - 1}\right| +\right.\\
&\quad \left. + \mathbf{i}\arg\left(k\pi + \sqrt{k^2\pi^2 - 1}\right)\right]\\
&= -\mathbf{i}\ln\left(k\pi + \sqrt{k^2\pi^2 - 1}\right) + 2n\pi
\end{aligned}
$$

e para $\xi = \xi_2$. procedendo do mesmo modo, tem-se

$$w = -\mathbf{i}\ln\left(k\pi - \sqrt{k^2\pi^2 - 1}\right) + 2n\pi$$

com $k, n \in \mathbb{Z}$. ∎

Exercício 2 *Observe que a função*

$$f(z) = z^{\operatorname{sen} z}$$

pode ser reescrita como

$$f(z) = e^{\operatorname{sen} z \operatorname{Ln} z}$$

e usando a regra da cadeia, tem-se que

$$
\begin{aligned}
f'(z) &= e^{\operatorname{sen} z \operatorname{Ln} z}\left[\operatorname{sen} z \operatorname{Ln} z\right]'\\
&= z^{\operatorname{sen} z}\left(\cos z \operatorname{Ln} z + \frac{\operatorname{sen} z}{z}\right)
\end{aligned}
$$

assim,

$$f'(\mathbf{i}) = \mathbf{i}^{\operatorname{sen}\mathbf{i}}\left(\cos\mathbf{i}\operatorname{Ln}\mathbf{i} + \frac{\operatorname{sen}\mathbf{i}}{\mathbf{i}}\right)$$

Lembrando que

$$\begin{aligned}\operatorname{sen}\mathbf{i} &= \frac{e^{-1}-e}{2\mathbf{i}}\\ &= \operatorname{senh}(1)\ \mathbf{i}\\ \cos\mathbf{i} &= \frac{e^{-1}+e}{2}\\ &= \cosh(1)\\ \operatorname{Ln}\mathbf{i} &= \ln|\mathbf{i}| + \mathbf{i}\operatorname{Arg}(\mathbf{i})\\ &= \frac{\pi}{2}\mathbf{i}\\ \mathbf{i}^{\operatorname{sen}\mathbf{i}} &= e^{\operatorname{sen}\mathbf{i}\operatorname{Ln}\mathbf{i}}\\ &= e^{\operatorname{senh}(1)\,\mathbf{i}\,\frac{\pi}{2}\mathbf{i}}\\ &= e^{-\operatorname{senh}(1)\frac{\pi}{2}}\end{aligned}$$

Assim,

$$f'(\mathbf{i}) = e^{-\operatorname{senh}(1)\frac{\pi}{2}}\left[\cosh(1)\ \frac{\pi}{2}\mathbf{i} + \operatorname{senh}(1)\right]$$

■

Exercício 3

a). *Uma parametrização possível para a curva C é dada por*

$$\begin{aligned}z(t) &= e^{\mathbf{i}t}\\ &= \cos t + \mathbf{i}\operatorname{sen} t,\ -\pi \le t \le 0\end{aligned}$$

Logo

$$\begin{aligned}A &= \int_C \left(x^2 - \mathbf{i}y^3\right) dz\\ &= \int_{-\pi}^{0}\left(\cos^2 t - \mathbf{i}\operatorname{sen}^3 t\right) z'(t)dt\\ &= \int_{-\pi}^{0}\left(\cos^2 t - \mathbf{i}\operatorname{sen}^3 t\right)(-\operatorname{sen} t + \mathbf{i}\ \cos t)\,dt\\ &= \int_{-\pi}^{0}\left(-\operatorname{sen} t\cos^2 t + \mathbf{i}\ \cos^3 t + \right.\\ &\quad \left. + \mathbf{i}\operatorname{sen}^3 t + \cos t\operatorname{sen}^3 t\right) dt\\ &= \int_{-\pi}^{0}\left[-\operatorname{sen} t\cos^2 t + \mathbf{i}\ \left(\cos t - \cos t\operatorname{sen}^2 t\right) + \right.\\ &\quad \left. + \mathbf{i}\left(\operatorname{sen} t - \operatorname{sen} t\cos^2 t\right) + \cos t\operatorname{sen}^3 t\right] dt\end{aligned}$$

$$\begin{aligned}A &= \int_{-\pi}^{0}\left[-(1+\mathbf{i})\operatorname{sen} t\cos^2 t + \left(\cos t + \operatorname{sen} t - \right.\right.\\ &\quad \left.\left. - \cos t\operatorname{sen}^2 t\right)\ \mathbf{i} + \cos t\operatorname{sen}^3 t\right] dt\\ &= \left|\frac{1}{3}(1+\mathbf{i})\cos^3 t + \left(\operatorname{sen} t - \cos t - \frac{1}{3}\operatorname{sen}^3 t\right)\mathbf{i} + \right.\\ &\quad \left. + \frac{1}{4}\operatorname{sen}^4 t\right|_{-\pi}^{0}\\ &= \frac{2}{3} - \frac{4}{3}\mathbf{i}\end{aligned}$$

□

b). *Tendo em vista que a função sob a integral é* ***analítica*** *em todo o plano complexo, a integral dada não depende do caminho e sim dos seus pontos inicial e final. Usando o* ***Teorema Fundamental para Integrais de Linha****, tem-se que*

$$\begin{aligned}\int_C e^z dz &= \int_0^{1+\pi\mathbf{i}} e^z dz\\ &= e^z\Big|_0^{1+\pi\mathbf{i}}\\ &= e^{1+\pi\mathbf{i}} - e^0\\ &= e(\cos\pi + \mathbf{i}\operatorname{sen}\pi) - 1\\ &= -e - 1\end{aligned}$$

■

Exercício 4 *Inicialmente observe que as singularidades da função sob a integral são os complexos tais que*

$$1 - e^z = 0$$

ou seja

$$\begin{aligned}z &= \operatorname{Ln} 1\\ &= \ln|1| + \mathbf{i}\operatorname{Arg}(1)\\ &= 0\end{aligned}$$

que está dentro da curva $|z| = 3$. Sabendo que

$$\begin{aligned}e^z &= \sum_{n=0}^{+\infty}\frac{z^n}{n!}\\ &= 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots\end{aligned}$$

*e usando a **primeira fórmula integral de Cauchy**, segue-se que*

$$
\begin{aligned}
\oint_C \frac{dz}{1-e^z} &= \oint_C \frac{dz}{-z-\frac{z^2}{2}-\cdots-\frac{z^n}{n!}-\cdots} \\
&= \oint_C \frac{dz}{z\left(-1-\frac{z}{2}-\cdots-\frac{z^{n-1}}{n!}-\cdots\right)} \\
&= \oint_C \frac{\frac{1}{-1-\frac{z}{2}-\cdots-\frac{z^{n-1}}{n!}-\cdots}}{z}\,dz \\
&= 2\pi\mathbf{i}\left.\frac{1}{-1-\frac{z}{2}-\cdots-\frac{z^{n-1}}{n!}-\cdots}\right|_{z=0} \\
&= -2\pi\mathbf{i}
\end{aligned}
$$

***Obs.:** Questão cancelada. O conteúdo sobre séries não foi abordado.* ■

Exercício 5 *Observe que*

$$
\begin{aligned}
\int_C (\operatorname{Re} z+1)\,\frac{f(z)}{z}dz &= \int_C \left(\frac{z+\bar{z}}{2}+1\right)\frac{f(z)}{z}dz \\
&= \int_C \left(\frac{z}{2}+\frac{\bar{z}}{2}\frac{z}{z}+1\right)\frac{f(z)}{z}dz \\
&= \int_C \left(\frac{z}{2}+\frac{|z|^2}{2z}+1\right)\frac{f(z)}{z}dz \\
&= \int_C \left(\frac{z}{2}+\frac{1}{2z}+1\right)\frac{f(z)}{z}dz \\
&= \int_C \left(\frac{f(z)}{2}+\frac{f(z)}{z}+\frac{f(z)}{2z^2}\right)dz
\end{aligned}
$$

*Uma vez que f é analítica na região $|z|<2$ e a curva C dada por $|z|=1$ está dentro desta região, segue-se do **Teorema de Cauchy-Gorsat** que*

$$\int_C \frac{f(z)}{2}dz = 0$$

*e das **fórmulas integrais de Cauchy**, que*

$$
\begin{aligned}
\int_C \frac{f(z)}{z}dz &= 2\pi\mathbf{i}f(0) \\
&= 2\pi\mathbf{i}
\end{aligned}
$$

$$
\begin{aligned}
\int_C \frac{f(z)}{2z^2}dz &= \pi\mathbf{i}f'(0) \\
&= \pi\mathbf{i}
\end{aligned}
$$

Portanto

$$\int_C (\operatorname{Re} z+1)\,\frac{f(z)}{z}dz = 3\pi\mathbf{i}$$

■