Universidade Federal do Vale do São Francisco
Engenharia Civil
Métodos Matemáticos

Prof^{o}. Edson

2 ^{o} Semestre

Gabarito 1ª Prova **2022**
Data: Sexta-feira, 19 de Maio **Turma E5**

Exercício 1 *Considere*

$$z = \frac{1}{\sqrt{7+24\mathbf{i}}}$$

e observe inicialmente, que

$$\begin{aligned} z &= \sqrt{\frac{1}{7+24\mathbf{i}}} \\ &= \left(\frac{1}{7+24\mathbf{i}}\frac{7-24\mathbf{i}}{7-24\mathbf{i}}\right)^{\frac{1}{2}} \\ &= \left(\frac{7-24\mathbf{i}}{625}\right)^{\frac{1}{2}} \\ &= \frac{(7-24\mathbf{i})^{\frac{1}{2}}}{25} \end{aligned}$$

Além disso, perceba que sendo

$$\begin{aligned} r &= |7-24\mathbf{i}| \\ &= \sqrt{625} \\ &= 25 \end{aligned}$$

e

$$\begin{aligned} \theta &= \mathbf{Arg}\,(7-24\mathbf{i}) \\ &= \operatorname{arctg}\left(\frac{-24}{7}\right) \end{aligned}$$

tem-se

$$\begin{aligned} 7-24\mathbf{i} &= r\,(\cos\theta + \mathbf{i}\operatorname{sen}\theta) \\ &= 25\,(\cos\theta + \mathbf{i}\operatorname{sen}\theta) \end{aligned}$$

Ou seja

$$\begin{aligned} z &= \frac{(7-24\mathbf{i})^{\frac{1}{2}}}{25} \\ &= \frac{[25\,(\cos\theta + \mathbf{i}\operatorname{sen}\theta)]^{\frac{1}{2}}}{25} \\ &= \frac{5\,(\cos\theta + \mathbf{i}\operatorname{sen}\theta)^{\frac{1}{2}}}{25} \\ &= \frac{(\cos\theta + \mathbf{i}\operatorname{sen}\theta)^{\frac{1}{2}}}{5} \\ &= \frac{\cos\left(\frac{\theta}{2}+k\pi\right) + \mathbf{i}\operatorname{sen}\left(\frac{\theta}{2}+k\pi\right)}{5}, \end{aligned}$$

com $k = 0, 1$. Portanto, tem-se duas possibilidades:

$$z_0 = \frac{1}{5}\left(\cos\frac{\theta}{2} + \mathbf{i}\operatorname{sen}\frac{\theta}{2}\right)$$

e

$$\begin{aligned} z_1 &= \frac{\cos\left(\frac{\theta}{2}+\pi\right) + \mathbf{i}\operatorname{sen}\left(\frac{\theta}{2}+\pi\right)}{5} \\ &= -\frac{1}{5}\left(\cos\frac{\theta}{2} + \mathbf{i}\operatorname{sen}\frac{\theta}{2}\right) \end{aligned}$$

Por fim, sabe-se que

$$\theta = \operatorname{arctg}\left(\frac{-24}{7}\right) \Leftrightarrow \operatorname{tg}\theta = -\frac{24}{7}$$

Segue-se disto, que

$$\operatorname{sen}\theta = -\frac{24}{25}$$

$$\cos\theta = \frac{7}{25}$$

e, além disto,

$$\begin{aligned}\operatorname{sen}\frac{\theta}{2} &= -\sqrt{\frac{1-\cos\theta}{2}}\\ &= -\sqrt{\frac{1-\frac{7}{25}}{2}}\\ &= -\sqrt{\frac{18}{25}\frac{1}{2}}\\ &= -\frac{3}{5}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{2}}\\ &= \sqrt{\frac{1+\frac{7}{25}}{2}}\\ &= \sqrt{\frac{32}{25}\frac{1}{2}}\\ &= \frac{4}{5}\end{aligned}$$

Então,

$$\begin{aligned}z_0 &= \frac{1}{5}\left(\cos\frac{\theta}{2} + \mathbf{i}\operatorname{sen}\frac{\theta}{2}\right)\\ &= \frac{1}{5}\left(\frac{4}{5} - \frac{3}{5}\right)\mathbf{i}\\ &= \frac{4}{25} - \frac{3}{25}\mathbf{i}\end{aligned}$$

e

$$z_1 = -\frac{4}{25} + \frac{3}{25}\mathbf{i}$$

□

Outro modo: *Observe que*

$$\begin{aligned}z &= \sqrt{\frac{1}{7+24\mathbf{i}}}\\ &= \left(\frac{1}{7+24\mathbf{i}}\frac{7-24\mathbf{i}}{7-24\mathbf{i}}\right)^{\frac{1}{2}}\\ &= \left(\frac{7-24\mathbf{i}}{625}\right)^{\frac{1}{2}}\\ &= \frac{(7-24\mathbf{i})^{\frac{1}{2}}}{25}\end{aligned}$$

Sejam $a, b \in R$ tais que

$$(7-24\mathbf{i})^{\frac{1}{2}} = a + b\mathbf{i},$$

ou seja,

$$\begin{aligned}7-24\mathbf{i} &= (a+b\mathbf{i})^2\\ &= (a^2-b^2) + 2ab\,\mathbf{i}\end{aligned}$$

Disto, segue-se que

$$\begin{cases} a^2-b^2 = 7\\ 2ab = -24\end{cases} \Leftrightarrow \begin{cases} a^2-b^2 = 7\\ ab = -12\end{cases}$$

Da segunda linha deste sistema, tem-se

$$b = -\frac{12}{a}$$

e substituindo-se este resultado na primeira equação, chega-se a

$$a^4 - 7a^2 - 144 = 0$$

Considerando

$$w = a^2,$$

a equação anterior, torna-se

$$w^2 - 7w - 144 = 0,$$

donde segue-se que $w = 16$. Ou seja

$$a^2 = 16 \Rightarrow a = \pm 4$$

Além disso, para $a = 4$, tem-se $b = -3$ e para $a = -4$, tem-se $b = 3$. Conclui-se, portanto, que

$$(7-24\mathbf{i})^{\frac{1}{2}} = \pm(4-3i)$$

e

$$\begin{aligned}z &= \sqrt{\frac{1}{7+24\mathbf{i}}}\\ &= \frac{(7-24\mathbf{i})^{\frac{1}{2}}}{25}\\ &= \pm\left(\frac{4}{25} - \frac{3}{25}i\right)\end{aligned}$$

■

Exercício 2 *Observe que*

$$\begin{aligned}z &= \mathbf{i}\left(\frac{1+\mathbf{i}}{1-\mathbf{i}}\right)^n\\ &= \mathbf{i}\left(\frac{1+\mathbf{i}}{1-\mathbf{i}}\frac{1+\mathbf{i}}{1+\mathbf{i}}\right)^n\\ &= \mathbf{i}\,(\mathbf{i})^n\\ &= \mathbf{i}^{n+1}\end{aligned}$$

Perceba ainda que

$$\begin{aligned}
\mathbf{i}^0 &= 1\\
\mathbf{i}^1 &= \mathbf{i}\\
\mathbf{i}^2 &= -1\\
\mathbf{i}^3 &= -\mathbf{i}\\
\mathbf{i}^4 &= 1\\
\mathbf{i}^5 &= \mathbf{i}\\
\mathbf{i}^6 &= -1\\
&\dots
\end{aligned}$$

Ou seja,

$$\begin{aligned}
\mathrm{Im}(z) = \mathrm{Im}(\mathbf{i}^{n+1}) &= 0\\
&\Rightarrow\\
n+1 &= 2k,\ k \in \mathbb{N}_*
\end{aligned}$$

Em outras palavras, $\mathrm{Im}(z) = 0$ *para todo* $n \in \mathbb{N}$*, tal que*

$$n = 2k-1,\ \ k \in \mathbb{N}_*$$

■

Exercício 3 *Sabe-se que*

$$z = \frac{1-\cos\alpha}{\operatorname{sen}\alpha\cos\alpha} + \frac{1-2\cos\alpha+2\operatorname{sen}\alpha}{\operatorname{sen}2\alpha}\mathbf{i},\ \ \alpha \in \left(0, \frac{\pi}{2}\right)$$

e $\mathrm{Arg}(z) = \frac{\pi}{4}$*. Ou seja*

$$\begin{aligned}
\operatorname{arctg}\left(\frac{\frac{1-2\cos\alpha+2\operatorname{sen}\alpha}{\operatorname{sen}2\alpha}}{\frac{1-\cos\alpha}{\operatorname{sen}\alpha\cos\alpha}}\right) &= \tfrac{\pi}{4} && \Rightarrow\\
\frac{1-2\cos\alpha+2\operatorname{sen}\alpha}{\operatorname{sen}2\alpha}\,\frac{\operatorname{sen}\alpha\cos\alpha}{1-\cos\alpha} &= \operatorname{tg}\tfrac{\pi}{4} && \Rightarrow\\
\frac{1-2\cos\alpha+2\operatorname{sen}\alpha}{2\operatorname{sen}\alpha\cos\alpha}\,\frac{\operatorname{sen}\alpha\cos\alpha}{1-\cos\alpha} &= 1 && \Rightarrow\\
\frac{1-2\cos\alpha+2\operatorname{sen}\alpha}{2\,(1-\cos\alpha)} &= 1 && \Rightarrow\\
1-2\cos\alpha+2\operatorname{sen}\alpha &= 2-2\cos\alpha && \Rightarrow\\
\operatorname{sen}\alpha &= \tfrac{1}{2} && \Rightarrow\\
\alpha &= \tfrac{\pi}{6}
\end{aligned}$$

■

Exercício 4 *Deseja-se verificar a continuidade da função*

$$f(z) = \frac{\mathrm{Re}(z)}{z+\mathbf{i}z} - 2z^2$$

em $z = e^{\frac{\mathbf{i}\pi}{4}}$*. Inicialmente observe que*

$$\begin{aligned}
z &= e^{\frac{\mathbf{i}\pi}{4}}\\
&= \cos\frac{\pi}{4} + \mathbf{i}\ \operatorname{sen}\frac{\pi}{4}\\
&= \frac{\sqrt{2}}{2} + \mathbf{i}\ \frac{\sqrt{2}}{2}
\end{aligned}$$

Perceba então, que

i) *Sendo* $z = e^{\frac{\mathbf{i}\pi}{4}}$*,*

$$\begin{aligned}
f(z) &= f\left(e^{\frac{\mathbf{i}\pi}{4}}\right)\\
&= \frac{\mathrm{Re}(z)}{z+\mathbf{i}z} - 2z^2\\
&= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} + \mathbf{i}\ \frac{\sqrt{2}}{2} + \mathbf{i}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}} - 2\mathbf{i}\\
&= -\frac{5}{2}\mathbf{i}
\end{aligned}$$

Ou seja, $f(e^{\frac{\mathbf{i}\pi}{4}})$ *existe.*

ii) *Considerando* $z = x + \mathbf{i}y$ *é possível reescrever* f *da seguinte maneira*

$$\begin{aligned}
f(z) &= \frac{x}{x+\mathbf{i}y+\mathbf{i}\,(x+\mathbf{i}y)} - 2\,(x+\mathbf{i}y)^2\\
&= \frac{x}{(x-y)+(x+y)\,\mathbf{i}} - 2\left(x^2-y^2+2xy\mathbf{i}\right)\\
&= \frac{x\left[(x-y)-(x+y)\,\mathbf{i}\right]}{(x-y)^2+(x+y)^2} - 2\left(x^2-y^2+2xy\mathbf{i}\right)\\
&= \frac{(x^2-xy)-(x^2+y^2)\,\mathbf{i}}{2\,(x^2+y^2)} - 2\left(x^2-y^2+2xy\mathbf{i}\right)\\
&= \frac{(x^2-xy)-4\,(x^4-y^4)}{2\,(x^2+y^2)} - \left(\frac{1}{2}+4xy\right)\mathbf{i}
\end{aligned}$$

ou seja

$$u(x,y) = \frac{(x^2-xy)-4\,(x^4-y^4)}{2\,(x^2+y^2)}$$

$$v(x,y) = -\left(\frac{1}{2}+4xy\right)$$

e

$$\lim_{z\to e^{\frac{\mathbf{i}\pi}{4}}} f(z) = \lim_{(x,y)\to\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)} u(x,y) + \mathbf{i} \lim_{(x,y)\to(0,0)} v(x,y)$$
$$= 0 - \mathbf{i}\frac{5}{2}$$
$$= -\frac{5}{2}\mathbf{i}$$

iii) *Por fim, segue-se dos itens anteriores que*

$$\lim_{z\to e^{\frac{\mathbf{i}\pi}{4}}} f(z) = f(e^{\frac{\mathbf{i}\pi}{4}})$$

Portanto, a função dada é contínua em $z = e^{\frac{\mathbf{i}\pi}{4}}$. ■

Exercício 5 *Deseja-se calcular o limite*

$$\lim_{z\to\mathbf{i}} \frac{\mathbf{i}z^3 - 1}{z - \mathbf{i}}$$

Para isto, perceba antes que

$$\mathbf{i}z^3 - 1 = (z - \mathbf{i})\left(\mathbf{i}z^2 - z - \mathbf{i}\right)$$

Logo,

$$\begin{aligned} \lim_{z\to\mathbf{i}} \frac{\mathbf{i}z^3 - 1}{z - \mathbf{i}} &= \lim_{z\to\mathbf{i}} \frac{(z - \mathbf{i})\left(\mathbf{i}z^2 - z - \mathbf{i}\right)}{z - \mathbf{i}} \\ &= \lim_{z\to\mathbf{i}} \left(\mathbf{i}z^2 - z - \mathbf{i}\right) \\ &= -3\mathbf{i} \end{aligned}$$

■

Exercício 6 (Desafio) *Sabe-se que*

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1}$$

para qualquer $z \in \mathbb{C}$. *Escolhendo*

$$\begin{aligned} z &= \cos\frac{\pi}{n} + \mathbf{i}\,\mathrm{sen}\,\frac{\pi}{n} \\ &= e^{\frac{\mathbf{i}2\pi}{n}} \end{aligned}$$

e substituindo na equação anterior, tem-se

$$1 + e^{\frac{\mathbf{i}2\pi}{n}} + e^{\frac{4\mathbf{i}\pi}{n}} + \cdots + e^{\frac{2(n-1)\mathbf{i}\pi}{n}} = \frac{e^{\mathbf{i}2\pi} - 1}{e^{\frac{\mathbf{i}2\pi}{n}} - 1} \Rightarrow$$
$$1 + e^{\frac{\mathbf{i}2\pi}{n}} + e^{\frac{4\mathbf{i}\pi}{n}} + \cdots + e^{\frac{2(n-1)\mathbf{i}\pi}{n}} = 0$$

Escrevendo na forma polar,

$$\left[1 + \cos\tfrac{2\pi}{n} + \cos\tfrac{4\pi}{n} + \cdots + \cos\tfrac{2(n-1)\pi}{n}\right] +$$
$$+\mathbf{i}\left[\mathrm{sen}\,\tfrac{2\pi}{n} + \mathrm{sen}\,\tfrac{4\pi}{n} + \mathrm{sen}\,\tfrac{6\pi}{n} + \cdots + \mathrm{sen}\,\tfrac{2(n-1)\pi}{n}\right] = 0$$

Ou seja,

$$\begin{aligned} 1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} &= 0 \\ \mathrm{sen}\,\frac{2\pi}{n} + \mathrm{sen}\,\frac{4\pi}{n} + \mathrm{sen}\,\frac{6\pi}{n} + \cdots + \mathrm{sen}\,\frac{2(n-1)\pi}{n} &= 0 \end{aligned}$$

e, por fim

$$S = 0$$

■