Universidade Federal do Vale do São Francisco
Engenharia Civil
Cálculo Diferencial e Integral III

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2^o Semestre

Gabarito 2ª Prova **2022**
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Exercício 1 *O volume que deseja-se calcular é dado pela integral*

$$\iiint_\Omega dx\,dy\,dz$$

sendo Ω a região do espaço dado por

$$\Omega : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 - x \\ \frac{1}{2}(1 - x - y) \le z \le 1 - x - y \end{cases}$$

Ou seja,

$$\begin{aligned} \iiint_\Omega dx\,dy\,dz &= \int_0^1 \int_0^{1-x} \int_{\frac{1}{2}(1-x-y)}^{1-x-y} dz\,dy\,dx \\ &= \int_0^1 \int_0^{1-x} z\Big|_{\frac{1}{2}(1-x-y)}^{1-x-y} dy\,dx \\ &= \frac{1}{2}\int_0^1 \int_0^{1-x} (1 - x - y)\,dy\,dx \\ &= \frac{1}{2}\int_0^1 \left(y - xy - \frac{1}{2}y^2\right)\Big|_0^{1-x} dx \\ &= \frac{1}{4}\int_0^1 (1-x)^2 dx \\ &= -\frac{1}{12}(1-x)^3\Big|_0^1 \\ &= \frac{1}{12} \end{aligned}$$

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Exercício 2 *Usando **coordenadas cilíndricas**, ou seja*

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

cujo jacobiano é

$$|J| = \left\|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right\| = r,$$

o conjunto Ω neste referencial, torna-se

$$\Omega_2 : \begin{cases} 0 \le r \le \sqrt{2} \\ 0 \le \theta \le 2\pi \\ 0 \le z \le r \end{cases}$$

Ou seja

$$\begin{aligned} \iiint_\Omega \left(x^2 + y^2\right) dx\,dy\,dz &= \iiint_{\Omega_2} r^2\,|J|\,dr\,d\theta\,dz \\ &= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^r r^3 dz\,d\theta\,dr \\ &= \int_0^{\sqrt{2}} \int_0^{2\pi} r^3 z\Big|_0^r d\theta\,dr \\ &= \int_0^{\sqrt{2}} \int_0^{2\pi} r^4 d\theta\,dr \\ &= \int_0^{\sqrt{2}} r^4\theta\Big|_0^{2\pi} dr \\ &= 2\pi\int_0^{\sqrt{2}} r^4 dr \\ &= 2\pi\,\frac{r^5}{5}\Big|_0^{\sqrt{2}} \\ &= \frac{8\sqrt{2}}{5}\pi \end{aligned}$$

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Exercício 3 *Usando **coordenadas esféricas**, ou seja*

$$\begin{cases} x = \rho\,\text{sen}\,\varphi\cos\theta \\ y = \rho\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = \rho\cos\varphi \end{cases}$$

cujo jacobiano é

$$|J| = \left\|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)}\right\| = \rho^2\text{sen}\,\varphi,$$

o conjunto Ω neste referencial, torna-se

$$\Omega_2 : \begin{cases} 0 \le \varphi \le \dfrac{\pi}{4} \\ 0 \le \rho \le 4\cos\varphi \\ 0 \le \theta \le 2\pi \end{cases}$$

Portanto,

$$\begin{aligned}
\iiint_{\Omega} dx\,dy\,dz &= \iiint_{\Omega_2} |J|\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{4\cos\varphi} \rho^2 \operatorname{sen}\varphi\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \frac{1}{3}\rho^3 \operatorname{sen}\varphi \Big|_0^{4\cos\varphi} d\theta\,d\varphi \\
&= \frac{64}{3}\int_0^{\frac{\pi}{4}}\int_0^{2\pi} \cos^3\varphi \operatorname{sen}\varphi\,d\theta\,d\varphi \\
&= \frac{64}{3}\int_0^{\frac{\pi}{4}} \cos^3\varphi \operatorname{sen}\varphi\,\theta \Big|_0^{2\pi} d\varphi \\
&= \frac{128\pi}{3}\int_0^{\frac{\pi}{4}} \cos^3\varphi \operatorname{sen}\varphi d\varphi \\
&= -\frac{32\pi}{3}\cos^4\varphi \Big|_0^{\frac{\pi}{4}} \\
&= 8\pi
\end{aligned}$$

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Exercício 4 *O* ***momento de inércia*** *da caixa em questão em torno do eixo z é dado por*

$$I = \iiint_{\Omega} r^2 \delta\,dx\,dy\,dz$$

sendo

$$\begin{aligned} r(x,y,z) &= \sqrt{x^2+y^2} \\ \delta(x,y,z) &= k,\ k \in \mathbb{R} \end{aligned}$$

e

$$\Omega : \begin{cases} -a \le x \le a \\ -a \le y \le a \\ 0 \le z \le h \end{cases}$$

Portanto,

$$\begin{aligned}
I &= \int_{-a}^{a}\int_{-a}^{a}\int_0^h k\left(x^2+y^2\right) dz\,dx\,dy \\
&= k\int_{-a}^{a}\int_{-a}^{a}\left(x^2+y^2\right) z\Big|_0^h dx\,dy \\
&= kh\int_{-a}^{a}\int_{-a}^{a}\left(x^2+y^2\right) dx\,dy \\
&= kh\int_{-a}^{a}\left(\frac{1}{3}x^3+y^2x\right)\Big|_{-a}^{a} dy \\
&= kh\int_{-a}^{a}\left(\frac{2}{3}a^3+2y^2a\right) dy \\
&= kh\left(\frac{2}{3}a^3y+\frac{2}{3}y^3a\right)\Big|_{-a}^{a} \\
&= \frac{8}{3}kha^4
\end{aligned}$$

Observe que

$$\begin{aligned} M &= \delta\,V(\Omega) \\ &= k\cdot 2a\cdot 2a\cdot h \\ &= 4ka^2h, \end{aligned}$$

Ou seja

$$\begin{aligned} I &= \frac{8}{3}kha^4 \\ &= \frac{2}{3}a^2\cdot M \end{aligned}$$

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Exercício 5

$$\iiint_{\Omega}\left(x^2y+3xyz\right) dx\,dy\,dz$$

Considere a seguinte mudança de variáveis

$$\varphi^{-1} : \begin{cases} u = x \\ v = xy \\ w = z \end{cases}$$

ou seja,

$$\varphi : \begin{cases} x = u \\ y = \dfrac{v}{u} \\ z = w \end{cases}$$

cujo jacobiano é dado por

$$|J| = \left\| \frac{\partial (x,y,z)}{\partial (u,v,w)} \right\| = \frac{1}{u}$$

Neste referencial, o conjunto Ω *torna-se*

$$\Omega_2 : \begin{cases} 1 \leq u \leq 2 \\ 0 \leq v \leq 2 \\ 0 \leq w \leq 1 \end{cases}$$

Assim,

$$\begin{aligned} A &= \iiint_{\Omega} \left(x^2 y + 3xyz\right) dx\, dy\, dz \\ &= \iiint_{\Omega_2} (uv + 3vw)\, |J|\, du\, dv\, dw \\ &= \int_1^2 \int_0^2 \int_0^1 (uv + 3vw)\, \frac{1}{u} dw\, dv\, du \\ &= \int_1^2 \int_0^2 \left(vw + \frac{3vw^2}{2u}\right) \Big|_0^1 dv\, du \\ &= \int_1^2 \int_0^2 \left(v + \frac{3v}{2u}\right) dv\, du \\ &= \int_1^2 \left(\frac{v^2}{2} + \frac{3v^2}{4u}\right) \Big|_0^2 du \\ &= \int_1^2 \left(2 + \frac{3}{u}\right) du \\ &= (2u + 3\ln|u|) \Big|_1^2 \\ &= 2 + 3\ln 2 \end{aligned}$$

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