Universidade Federal do Vale do São Francisco
Engenharia Civil
Cálculo Diferencial e Integral III

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1^{o} Semestre

Gabarito 2ª Prova **2021**
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Exercício 1 *Desejamos calcular a integral*

$$A = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy\, dz\, dx$$

Observe que, neste caso, o domínio de integração é o conjunto

$$\Omega : \begin{cases} 0 \leq x \leq 4 \\ 0 \leq y \leq \sqrt{16 - x^2 - z^2} \\ 0 \leq z \leq \sqrt{16 - x^2} \end{cases}$$

Assim, usando ***coordenadas esféricas****,*

$$\begin{cases} x = \rho \operatorname{sen} \varphi \cos \theta \\ y = \rho \operatorname{sen} \varphi \operatorname{sen} \theta \\ z = \rho \cos \theta \end{cases}$$

cujo jacobiano é dado por

$$|J| = \left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} \right| = \rho^2 \operatorname{sen} \varphi$$

O conjunto Ω torna-se

$$\Omega_2 : \begin{cases} 0 \leq \varphi \leq \dfrac{\pi}{2} \\ 0 \leq \rho \leq 4 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \end{cases}$$

Portanto,

$$\begin{aligned} A &= \iiint_{\Omega_2} |J|\, d\rho\, d\varphi\, d\theta \\ &= \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^2 \operatorname{sen} \varphi d\theta\, d\varphi\, d\rho \\ &= \int_0^4 \int_0^{\frac{\pi}{2}} \rho^2 \operatorname{sen} \varphi\, \theta \Big|_0^{\frac{\pi}{2}} d\varphi\, d\rho \\ &= \frac{\pi}{2} \int_0^4 \int_0^{\frac{\pi}{2}} \rho^2 \operatorname{sen} \varphi\, d\varphi\, d\rho \\ &= -\frac{\pi}{2} \int_0^4 \rho^2 \cos\, \varphi \Big|_0^{\frac{\pi}{2}} d\rho \\ &= -\frac{\pi}{2} \int_0^4 \rho^2 \cos\, \varphi \Big|_0^{\frac{\pi}{2}} d\rho \\ &= \frac{\pi}{2} \int_0^4 \rho^2 d\rho \\ &= \frac{\pi}{2} \frac{\rho^3}{3} \Big|_0^4 \\ &= \frac{32\pi}{3} \end{aligned}$$

□

(Outro modo:) *Observe que a integral em questão consiste no cálculo do volume de um sólido que é a partede um esfera de raio 4 que encontr-se no primeiro octante, ou seja*

$$\begin{aligned} A &= \frac{1}{8} \mathrm{Vol}(\textit{esfera}) \\ &= \frac{1}{8} \frac{4}{3} \pi \cdot 4^3 \\ &= \frac{32\pi}{3} \end{aligned}$$

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Exercício 2 *Considere a seguinte* ***mudança de variáveis***

$$\varphi^{-1} : \begin{cases} u = x + 2y \\ v = x - z \\ w = 2y - z \end{cases}$$

Ou seja,

$$\varphi : \begin{cases} x = \dfrac{u + v - w}{2} \\ y = \dfrac{u - v + w}{4} \\ z = \dfrac{u - v - w}{2} \end{cases}$$

cujo jacobiano é dado por

$$|J| = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \frac{1}{4}$$

e neste referencial o conjunto Ω torna-se

$$\Omega_2 : \begin{cases} 0 \leq u \leq 2 \\ 0 \leq v \leq 2 \\ 0 \leq w \leq 3 \end{cases}$$

Portanto,

$$\begin{aligned} I &= \iiint_{\Omega} yz\,dx\,dy\,dz \\ &= \iiint_{\Omega_2} \left(\frac{u-v+w}{4}\right)\left(\frac{u-v-w}{2}\right)|J|\,du\,dv\,dw \\ &= \frac{1}{32}\int_0^2\int_0^2\int_0^3 (u-v+w)(u-v-w)\,dw\,dv\,du \\ &= -\frac{7}{8} \end{aligned}$$

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Exercício 3 *Usando **coordenadas esféricas**,*

$$\begin{cases} x = \rho\,\mathrm{sen}\,\varphi\cos\theta \\ y = \rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta \\ z = \rho\cos\theta \end{cases}$$

cujo jacobiano é dado por

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2\mathrm{sen}\,\varphi$$

O conjunto Ω dado na questão, torna-se, neste referencial,

$$\Omega_2 : \begin{cases} \dfrac{\pi}{6} \leq \varphi \leq \dfrac{\pi}{3} \\ 0 \leq \rho \leq 4 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

Portanto,

$$\begin{aligned} V &= \iiint_{\Omega_2} |J|\,d\rho\,d\varphi\,d\theta \\ &= \int_0^4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^{2\pi} \rho^2\mathrm{sen}\,\varphi d\theta\,d\varphi\,d\rho \\ &= \int_0^4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \rho^2\mathrm{sen}\,\varphi\,\theta\Big|_0^{2\pi} d\varphi\,d\rho \\ &= 2\pi\int_0^4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \rho^2\mathrm{sen}\,\varphi\,d\varphi\,d\rho \\ &= -2\pi\int_0^4 \rho^2\cos\,\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\rho \\ &= \pi\left(\sqrt{3}-1\right)\int_0^4 \rho^2 d\rho \\ &= \frac{64\pi}{3}\left(\sqrt{3}-1\right) \end{aligned}$$

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Exercício 4 *Usando **coordenadas cilíndricas**,*

$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ z = z \end{cases}$$

cujo jacobiano é dado por

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

O conjunto Ω dado na questão, torna-se, neste referencial,

$$\Omega_2 : \begin{cases} -\dfrac{\pi}{4} \leq \theta \leq \dfrac{3\pi}{4} \\ 0 \leq r \leq 1 \\ 0 \leq z \leq r\,(\cos\theta + \mathrm{sen}\,\theta) \end{cases}$$

Portanto,

$$\begin{aligned} V &= \iiint_{\Omega_2} |J|\,dr\,d\theta\,dz \\ &= \int_0^1\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{r(\cos\theta+\mathrm{sen}\,\theta)} rdz\,d\theta\,dr \\ &= \int_0^1\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} rz\Big|_0^{r(\cos\theta+\mathrm{sen}\,\theta)} d\theta\,dr \\ &= \int_0^1\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} r^2\,(\cos\theta+\mathrm{sen}\,\theta)\,d\theta\,dr \\ &= \int_0^1 r^2\,(\mathrm{sen}\,\theta-\cos\theta)\Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} dr \\ &= 2\sqrt{2}\int_0^1 r^2 dr \\ &= \frac{2\sqrt{2}}{3} \end{aligned}$$

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Exercício 5 *A massa do sólido em questão é dada por*

$$\begin{aligned}
\int_0^2\int_0^2\int_0^2 \delta dx\,dy\,dz &= \int_0^2\int_0^2\int_0^2 (x+y+z+1)\,dx\,dy\,dz \\
&= \int_0^2\int_0^2 (2y+2z+4)\,dy\,dz \\
&= \int_0^2 (4z+12)\,dz \\
&= 32
\end{aligned}$$

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