Universidade Federal do Vale do São Francisco
Engenharia Civil
Cálculo Diferencial e Integral III

Prof^{o}. Edson

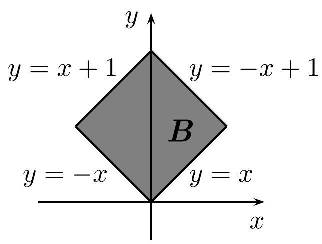
1^{o} Semestre

Gabarito Prova Final **2009**
Data: Segunda-feira, 15 de Junho **Turma E3**

Exercício 1 *Observe que*

$$\iint_B \sqrt[3]{y^2 - x^2}dxdy = \iint_B \sqrt[3]{(y-x)(y+x)}dxdy$$

e o conjunto B possui a seguinte representação gráfica



Fazendo a mudança de variáveis

$$\begin{cases} u = x + y \\ v = -x + y \end{cases}$$

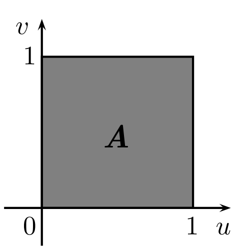
teremos

$$\begin{cases} x = \dfrac{u - v}{2} \\ u = \dfrac{u + v}{2} \end{cases}$$

cujo jacobiano é

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \left|\frac{1}{4} + \frac{1}{4}\right| = \frac{1}{2}$$

e, neste novo sistema de variáveis o conjunto B será transformado no conjunto A que possui o seguinte gráfico

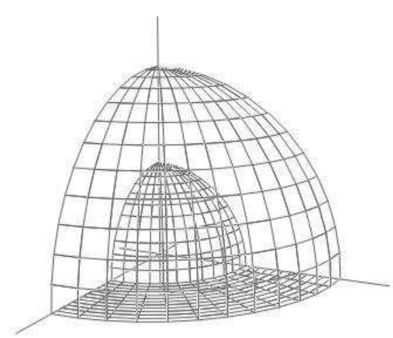


Ou seja

$$\begin{aligned} \iint_B \sqrt[3]{y^2 - x^2}dxdy &= \iint_A \frac{1}{2}\sqrt[3]{uv}dudv \\ &= \frac{1}{2}\int_0^1\int_0^1 \sqrt[3]{u}\sqrt[3]{v}dudv \\ &= \frac{1}{2}\int_0^1 \frac{3}{4}\sqrt[3]{v}\sqrt[3]{u^4}\Big|_0^1 dv \\ &= \frac{1}{2}\frac{3}{4}\int_0^1 \sqrt[3]{v}dv \\ &= \frac{9}{32} \end{aligned}$$

■

Exercício 2 *Sendo B a região do espaço delimitada pelas esferas $x^2 + y^2 + z^2 = 1$ e $x^2 + y^2 + z^2 = 4$ no primeiro octante, teremos o seguinte esboço da mesma*



Usando coordenadas esféricas, ou seja, tomando

$$\begin{cases} x = r\,\text{sen}\,\varphi\cos\theta \\ y = r\,\text{sen}\,\varphi\,\text{sen}\,\theta \\ z = r\cos\varphi \end{cases}$$

o conjunto B torna-se

$$B : \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 \leq \varphi \leq \dfrac{\pi}{2} \end{cases}$$

e, o jacobiano desta mudança de variáveis é

$$\left| \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} \right| = r^2 \operatorname{sen}\varphi$$

Donde segue-se que

$$\begin{aligned}
\iiint_B z dxdydz &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 r\cos\varphi r^2 \operatorname{sen}\varphi dr d\theta d\varphi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 r^3 \cos\varphi \operatorname{sen}\varphi dr d\theta d\varphi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^4}{4} \cos\varphi \operatorname{sen}\varphi \Big|_1^2 d\theta d\varphi \\
&= \frac{15}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{sen}\varphi d\theta d\varphi \\
&= \frac{15}{4} \int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{sen}\varphi \theta \Big|_0^{\frac{\pi}{2}} d\varphi \\
&= \frac{15\pi}{8} \int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{sen}\varphi d\varphi
\end{aligned}$$

Tomando

$$u = \operatorname{sen}\varphi$$

e observando que

$$du = \cos\varphi d\varphi$$

$$\varphi = 0 \Rightarrow u = 0$$

$$\varphi = \frac{\pi}{2} \Rightarrow u = 1$$

Segue-se que

$$\begin{aligned}
\iiint_B z dxdydz &= \frac{15\pi}{8} \int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{sen}\varphi d\varphi \\
&= \frac{15\pi}{8} \int_0^1 u du \\
&= \frac{15\pi}{8} \frac{u^2}{2} \Big|_0^1 \\
&= \frac{15\pi}{16}
\end{aligned}$$

■

Exercício 3 *Sendo γ a metade direita do círculo*

$$x^2 + y^2 = 16,$$

podemos parametrizar a curva γ da seguinte forma

$$\begin{cases} x(t) = 4\cos t \\ y(t) = 4\operatorname{sen} t \end{cases}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

De onde segue-se que

$$\begin{aligned}
ds &= \sqrt{x'(t)^2 + y'(t)^2} dt \\
&= \sqrt{16\cos^2 t + 16\operatorname{sen}^2 t} dt \\
&= 4dt
\end{aligned}$$

Logo,

$$\begin{aligned}
\int_\gamma xy^4 ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos t (4\operatorname{sen} t)^4 4dt \\
&= 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \operatorname{sen}^4 t dt
\end{aligned}$$

Tome

$$u = \operatorname{sen} t$$

e observe que

$$du = \cos t \, dt$$

$$t = -\tfrac{\pi}{2} \Rightarrow u = -1$$

$$t = \tfrac{\pi}{2} \Rightarrow u = 1$$

Então

$$\begin{aligned}
\int_\gamma xy^4 ds &= 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \operatorname{sen}^4 t dt \\
&= 4^6 \int_{-1}^1 u^4 du \\
&= \frac{4^6}{5} u^5 \Big|_{-1}^1 \\
&= \frac{2 \cdot 4^6}{5}
\end{aligned}$$

■

Exercício 4 *Sabemos que o fluxo do campo vetorial*

$$\mathbf{F}(x,y,z) = e^{-y}\mathbf{i} - y\mathbf{j} + xsenz\mathbf{k}$$

através da superfície

$$\varphi : \begin{cases} x(u,v) = 2\cos v \\ y(u,v) = \operatorname{sen} v \\ z(u,v) = u \end{cases}$$

onde $(u,v) \in K$, *com*

$$K : \begin{cases} 0 \leq u \leq 5 \\ 0 \leq v \leq 2\pi \end{cases}$$

na direção do vetor normal $\mathbf{n}$, *é dado por*

$$\iint_{\varphi} \mathbf{F} \cdot \mathbf{n} dS$$

Para resolvermos esta integral de superfície, precisamos antes calcular:

$$\frac{\partial\varphi}{\partial u}(u,v) = (0,0,1)$$

$$\frac{\partial\varphi}{\partial v}(u,v) = (-2\operatorname{sen} v, \cos v, 0)$$

donde segue-se que

$$\frac{\partial\varphi}{\partial u}(u,v) \times \frac{\partial\varphi}{\partial v}(u,v) = (-\cos v, -2\operatorname{sen} v, 0)$$

Com isto, temos então, que

$$\iint_{\varphi} \mathbf{F} \cdot \mathbf{n} dS = \iint_{K} \mathbf{F}(\varphi(u,v)).\left(\frac{\partial\varphi}{\partial u}(u,v) \times \frac{\partial\varphi}{\partial v}\right) dudv$$

$$= \iint_{K} \mathbf{F}(2\cos v, \operatorname{sen} v, u)\cdot (-\cos v, -2\operatorname{sen} v, 0)\, dudv$$

$$= \iint_{K} (e^{-\operatorname{sen} v}, -\operatorname{sen} v, 2\cos v \operatorname{sen} u)\cdot (-\cos v, -2\operatorname{sen} v, 0)\, dudv$$

$$= \iint_{K} \left(-\cos v e^{-\operatorname{sen} v} + 2\operatorname{sen}^2 v\right) dudv$$

$$= \int_0^{2\pi}\int_0^5 \left(-\cos v e^{-\operatorname{sen} v} + 2\operatorname{sen}^2 v\right) dudv$$

$$= \int_0^{2\pi} \left(-\cos v e^{-\operatorname{sen} v} + 2\operatorname{sen}^2 v\right) u\Big|_0^5 dv$$

$$= 5\int_0^{2\pi} \left(-\cos v e^{-\operatorname{sen} v} + 2\operatorname{sen}^2 v\right) dv$$

$$= 5\left(e^{-\operatorname{sen} v} + v - \frac{\operatorname{sen} 2v}{2}\right)\Big|_0^{2\pi}$$

$$= 10\pi$$

■

Exercício 5 *A superfície dada no problema possui a seguinte parametrização*

$$\varphi : \begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = 1-u-v \end{cases}$$

com $(u,v) \in K$, *onde*

$$K : \begin{cases} 0 \leq u \leq 1 \\ 0 \leq v \leq 1-u \end{cases}$$

*O **Teorema de Stokes** afirma que*

$$\oint_{\Gamma} \mathbf{F} \cdot d\Gamma = \iint_{\varphi} \operatorname{rot} \mathbf{F} \cdot \mathbf{n} dS$$

Observe porém que

$$\operatorname{rot}\mathbf{F} = (-y, -z, -x)$$

e, além disto temos que

$$\frac{\partial\varphi}{\partial u}(u,v) = (1,0,-1)$$
$$\frac{\partial\varphi}{\partial v}(u,v) = (0,1,-1)$$

donde segue-se que

$$\frac{\partial\varphi}{\partial u}(u,v) \times \frac{\partial\varphi}{\partial v}(u,v) = (1,1,1)$$

Portanto,

$$\oint_\Gamma \mathbf{F}\cdot d\Gamma = \iint_\varphi \operatorname{rot}\mathbf{F}\cdot\mathbf{n}dS$$
$$= \int_0^1\int_0^{1-u} \operatorname{rot}\mathbf{F}(\varphi(u,v))\cdot(1,1,1)dvdu$$
$$= \int_0^1\int_0^{1-u} \operatorname{rot}\mathbf{F}(u,v,1-u-v)\cdot(1,1,1)dvdu$$
$$= \int_0^1\int_0^{1-u} (-v, u+v-1, -u)\cdot(1,1,1)dvdu$$
$$= \int_0^1\int_0^{1-u} (-1)dvdu$$
$$= -\frac{1}{2}$$

■