Universidade Federal do Vale do São Francisco
Engenharia Civil
Cálculo Diferencial e Integral III

Prof^{o}. Edson

1^{o} Semestre

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Data: Quarta-feira, 11 de Julho **Turma M3**

Exercício 1

a). Considere a seguinte mudança de variável

$$\begin{cases} u = y - x^2 \\ v = x \end{cases}$$

teremos com isto

$$\begin{cases} x = v \\ y = u - v^2 \end{cases}$$

e o jacobiano desta mudança será

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 0 & 1 \\ 1 & -2v \end{vmatrix} = 1$$

e além disso

$$\begin{array}{lcl} x = 0 & \Rightarrow & v = 0 \\ y = 1 + x^2 & \Rightarrow & u = 1 \\ y = 1 + x^2 & \Rightarrow & u = 2 \\ y = x + x^2 & \Rightarrow & u = v \end{array}$$

ou seja,

$$\begin{aligned} \iint_B \frac{e^{y-x^2}}{y-x^2}dxdy &= \int_1^2\int_0^u \frac{e^u}{u}dvdu \\ &= \int_1^2 \tfrac{e^u}{u}\ v\big|_0^u\, du \\ &= \int_1^2 e^u du \\ &= e^u\big|_1^2 \\ &= e^2 - e \end{aligned}$$

□

b). Observe que

$$4x^2 + y^2 = 1 \Leftrightarrow (2x)^2 + y^2 = 1$$

então, usando coordenadas polares, podemos representar a região

$$B = \left\{ (x,y) \in R^2 \big| \, 4x^2 + y^2 \leq 1, y \geq 0 \right\}$$

da seguinte maneira

$$\begin{cases} 2x = r\cos\,\theta \\ y = r\mathrm{sen}\,\theta \end{cases}$$

com

$$\begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \pi \end{array}$$

e

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{1}{2}\cos\,\theta & -\frac{1}{2}r\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & r\cos\,\theta \end{vmatrix} = \frac{1}{2}r$$

Logo

$$\begin{aligned} \iint_B x^2dxdy &= \int_0^1\int_0^\pi \tfrac{1}{4}r^2\cos^2\theta\tfrac{1}{2}rd\theta dr \\ &= \tfrac{1}{8}\int_0^1\int_0^\pi r^3\cos^2\theta d\theta dr \\ &= \tfrac{1}{8}\int_0^1 r^3\ \left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)\Big|_0^\pi \\ &= \tfrac{\pi}{16}\int_0^1 r^3dr \\ &= \tfrac{\pi}{16}\ \tfrac{r^4}{4}\Big|_0^1 \\ &= \tfrac{\pi}{64} \end{aligned}$$

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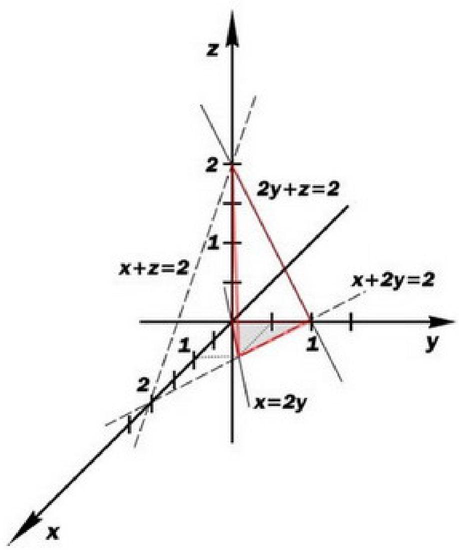
Exercício 2 *Esboçando a região dada obtemos o seguinte desenho:*



Desta forma, o tetraedro em questão pode ser representado pelo conjunto

$$R = \left\{(x,y,z) \in \mathbb{R}^3; 0 \leq x \leq 1, \tfrac{x}{2} \leq y \leq \tfrac{2-x}{2}, \; 0 \leq z \leq 2-x-2y\right\}$$

e seu volume pode ser obtido da seguinte forma

$$\begin{aligned} V &= \iiint_R dxdydz \\ &= \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_0^{2-x-2y} dzdydx \\ &= \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2-x-2y)\, dydx \\ &= \int_0^1 \left(2y - xy - y^2\right)\Big|_{\frac{x}{2}}^{\frac{2-x}{2}} dx \\ &= \int_0^1 \left(1 - 2x + x^2\right) dx \\ &= x - x^2 + \tfrac{1}{3}x^3\Big|_0^1 \\ &= \tfrac{1}{3} \end{aligned}$$

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Exercício 3 *A massa do cone em questão pode ser obtida através da seguinte integral*

$$M = \iiint_C \delta(x,y,z)dxdzdy$$

onde $\delta(x,y,z)$ é a densidade do cone C no ponto de coordenadas (x,y,z).
Como a densidade no ponto (x,y,z) é proporcional ao quadrado da distância deste ponto ao eixo z, segue-se que

$$\delta(x,y,z) = k(x^2+y^2), k \in \mathbb{R}$$

Assim, usando coordenadas cilíndricas, ou seja

$$\begin{cases} x = r\cos\theta & \qquad 0 \leq \theta \leq 2\pi \\ y = r\text{sen }\theta \quad \textit{onde} & \qquad 0 \leq r \leq 1 \\ z = z & \sqrt{x^2+y^2} \leq z \leq 1 \end{cases}$$

e

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

Desta forma, a massa do cone será

$$M = \int_0^{2\pi}\int_0^1\int_{\sqrt{x^2+y^2}}^1 k(x^2+y^2)rdzdrd\theta$$

Resolvendo esta integral, teremos

$$\begin{aligned} M &= k\int_0^{2\pi}\int_0^1\int_{\sqrt{x^2+y^2}}^1 (x^2+y^2)rdzdrd\theta \\ &= k\int_0^{2\pi}\int_0^1\int_r^1 r^3 dzdrd\theta \\ &= k\int_0^{2\pi}\int_0^1 r^3 z\Big|_r^1 drd\theta \\ &= k\int_0^{2\pi}\int_0^1 (r^3 - r^4)drd\theta \\ &= k\int_0^{2\pi} \tfrac{1}{4}r^4 - \tfrac{1}{5}r^5\Big|_0^1 d\theta \\ &= \tfrac{k}{20}\int_0^{2\pi} d\theta \\ &= \tfrac{k\pi}{10} \end{aligned}$$

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Exercício 4 *Observe que*

$$4x^2 + y^2 = 9 \Leftrightarrow \left(\frac{2x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Então, a curva γ pode ser parametrizada da seguinte forma

$$\gamma : \begin{cases} \frac{2x}{3} = \cos\theta \\ \frac{y}{3} = \text{sen }\theta \end{cases} \quad \textit{com } 0 \leq \theta \leq 2\pi$$

Desta forma, se considerarmos

$$A = \int_\gamma \frac{-y}{4x^2+y^2}dx + \frac{x}{4x^2+y^2}dy$$

teremos que

$$\begin{aligned} A &= \int_0^{2\pi} \tfrac{-3\text{sen }\theta}{9}\tfrac{-3\text{sen }\theta}{2}d\theta + \tfrac{3\cos\theta}{2}\tfrac{3\cos\theta}{9}d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} \left(\text{sen}^2\ \theta + \cos^2\theta\right) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} d\theta \\ &= \pi \end{aligned}$$

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Exercício 5

a).

$$\oint_\gamma \frac{\partial g}{\partial n} d\gamma = \oint_\gamma \nabla g \cdot n d\gamma$$

como a curva γ e a região R satisfazem todas as condições do Teorema de Green, segue-se então que

$$\begin{aligned} \oint_\gamma \tfrac{\partial g}{\partial n} d\gamma &= \oint_\gamma \nabla g \cdot n d\gamma \\ &= \iint_R div\left(\nabla g\right) dxdy \\ &= \iint_R div\left(\tfrac{\partial g}{\partial x}, \tfrac{\partial g}{\partial y}\right) dxdy \\ &= \iint_R \left(\tfrac{\partial^2 g}{\partial x^2} + \tfrac{\partial^2 g}{\partial y^2}\right) dxdy \\ &= \iint_R \nabla^2 g dxdy \end{aligned}$$

□

b).

$$\oint_\gamma f\frac{\partial g}{\partial n} d\gamma = \oint_\gamma f\nabla g \cdot n d\gamma$$

como a curva γ e a região R satisfazem todas as condições do Teorema de Green, segue-se então que

$$\begin{aligned} \oint_\gamma f\tfrac{\partial g}{\partial n} d\gamma &= \oint_\gamma f\nabla g \cdot n d\gamma \\ &= \iint_R div\left(f\nabla g\right) dxdy \\ &= \iint_R div\left(f\tfrac{\partial g}{\partial x}, f\tfrac{\partial g}{\partial y}\right) dxdy \\ &= \iint_R \left(\tfrac{\partial f}{\partial x}\tfrac{\partial g}{\partial x} + f\tfrac{\partial^2 g}{\partial x^2} + \right. \\ &\quad \left. + \tfrac{\partial f}{\partial y}\tfrac{\partial g}{\partial y} + f\tfrac{\partial^2 g}{\partial y^2}\right) dxdy \\ &= \iint_R \left[\left(\tfrac{\partial f}{\partial x}\tfrac{\partial g}{\partial x} + \tfrac{\partial f}{\partial y}\tfrac{\partial g}{\partial y}\right) + \right. \\ &\quad \left. + f\left(\tfrac{\partial^2 g}{\partial x^2} + \tfrac{\partial^2 g}{\partial y^2}\right)\right] dxdy \\ &= \iint_R \left[\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right) \cdot \left(\tfrac{\partial g}{\partial x}, \tfrac{\partial g}{\partial y}\right) + \right. \\ &\quad \left. + f\left(\tfrac{\partial^2 g}{\partial x^2} + \tfrac{\partial^2 g}{\partial y^2}\right)\right] dxdy \\ &= \iint_R \left(\nabla f \cdot \nabla g + f\nabla^2 g\right) dxdy \end{aligned}$$

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