

Universidade Federal do Vale do São Francisco  
Engenharia Civil  
Cálculo Diferencial e Integral III

Prof<sup>o</sup>. Edson

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Gabarito 3<sup>a</sup> Prova  
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Prof<sup>o</sup>. Edson

**Exercício 1** Da definição de derivada direcional segue-se que

$$\frac{\partial \varphi}{\partial \mathbf{n}} = \nabla \varphi \cdot \mathbf{n}$$

Assim,

$$\iint_S \frac{\partial \varphi}{\partial \mathbf{n}} dS = \iint_S \nabla \varphi \cdot \mathbf{n} dS$$

Usando a definição de integral de superfície, teremos

$$\begin{aligned} \iint_S \nabla \varphi \cdot \mathbf{n} dS &= \iint_S \nabla \varphi \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} \|\mathbf{n}\| dudv \\ &= \iint_S \nabla \varphi \cdot \left( \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) dudv \quad (1) \\ &= \iiint_{\dot{S}} \operatorname{div}(\nabla \varphi) dx dy dz \end{aligned}$$

Observe agora que

$$\nabla \varphi = \left( \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$

e disto, temos que

$$\operatorname{div}(\nabla \varphi) = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

Por outro lado temos também que

$$\varphi \nabla \varphi = \left( \varphi \frac{\partial \varphi}{\partial x}, \varphi \frac{\partial \varphi}{\partial y}, \varphi \frac{\partial \varphi}{\partial z} \right)$$

donde

$$\begin{aligned} \operatorname{div}(\varphi \nabla \varphi) &= \left( \frac{\partial \varphi}{\partial x} \right)^2 + \varphi \frac{\partial^2 \varphi}{\partial x^2} + \left( \frac{\partial \varphi}{\partial y} \right)^2 + \varphi \frac{\partial^2 \varphi}{\partial y^2} + \\ &\quad + \left( \frac{\partial \varphi}{\partial z} \right)^2 + \varphi \frac{\partial^2 \varphi}{\partial z^2} \\ &= \left[ \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right] + \\ &\quad + \varphi \left[ \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right] \\ &= \|\nabla \varphi\|^2 + \varphi \operatorname{div}(\nabla \varphi) \end{aligned}$$

Como

$$\|\nabla \varphi\|^2 = 4\varphi$$

$$\operatorname{div}(\varphi \nabla \varphi) = 10\varphi$$

segue-se que

$$10\varphi = 4\varphi + \varphi \operatorname{div}(\nabla \varphi) \Leftrightarrow \operatorname{div}(\nabla \varphi) = 6$$

Voltando para a equação (1) teremos

$$\begin{aligned} \iint_S \nabla \varphi \cdot \mathbf{n} dS &= \iiint_{\dot{S}} \operatorname{div}(\nabla \varphi) dx dy dz \\ &= 6 \iiint_{\dot{S}} dx dy dz \\ &= 8\pi \end{aligned}$$

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**Exercício 2**

a).

$$\oint_{\gamma} \frac{\partial g}{\partial \mathbf{n}} d\gamma = \oint_{\gamma} \nabla g \cdot \mathbf{n} d\gamma$$

como a curva  $\gamma$  e a região  $R$  satisfazem todas as condições do Teorema de Green, segue-se então que

$$\begin{aligned} \oint_{\gamma} \frac{\partial g}{\partial \mathbf{n}} d\gamma &= \oint_{\gamma} \nabla g \cdot \mathbf{n} d\gamma \\ &= \iint_R \operatorname{div}(\nabla g) dx dy \\ &= \iint_R \operatorname{div} \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) dx dy \\ &= \iint_R \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) dx dy \\ &= \iint_R \nabla^2 g dx dy \end{aligned}$$

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b).

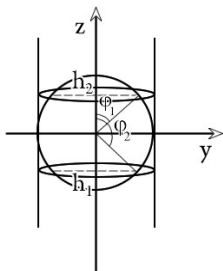
$$\oint_{\gamma} f \frac{\partial g}{\partial n} d\gamma = \oint_{\gamma} f \nabla g \cdot n d\gamma$$

como a curva  $\gamma$  e a região  $R$  satisfazem todas as condições do Teorema de Green, segue-se então que

$$\begin{aligned} \oint_{\gamma} f \frac{\partial g}{\partial n} d\gamma &= \oint_{\gamma} f \nabla g \cdot n d\gamma \\ &= \iint_R \operatorname{div} (f \nabla g) dx dy \\ &= \iint_R \operatorname{div} \left( f \frac{\partial g}{\partial x}, f \frac{\partial g}{\partial y} \right) dx dy \\ &= \iint_R \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial y^2} \right) dx dy \\ &= \iint_R \left[ \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \right) + f \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) \right] dx dy \\ &= \iint_R \left[ \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) + f \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) \right] dx dy \\ &= \iint_R (\nabla f \cdot \nabla g + f \nabla^2 g) dx dy \end{aligned}$$

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**Exercício 3** Sejam  $h_1$  e  $h_2$  as alturas em que os planos cortam a esfera.



A área  $A_c$  do cilindro que está entre os planos é

$$A_c = 2\pi r(h_2 - h_1)$$

Podemos parametrizar a porção  $\sigma$  da superfície da esfera que está entre os planos, da seguinte forma:

$$\sigma : \begin{cases} x = r \operatorname{sen} \theta \operatorname{sen} \varphi \\ y = r \operatorname{cos} \theta \operatorname{sen} \varphi \\ z = r \operatorname{cos} \varphi \end{cases} \quad \text{onde} \quad \begin{cases} 0 \leq \theta \leq 2\pi \\ \varphi_1 \leq \varphi \leq \varphi_2 \end{cases}$$

onde

$$\cos \varphi_1 = \frac{h_2}{r} \quad \text{e} \quad \cos \varphi_2 = \frac{h_1}{r}$$

Assim, temos que

$$\frac{\partial \sigma}{\partial \theta} = (r \operatorname{cos} \theta \operatorname{sen} \varphi, r \operatorname{sen} \theta \operatorname{sen} \varphi, 0)$$

$$\frac{\partial \sigma}{\partial \varphi} = (r \operatorname{sen} \theta \operatorname{cos} \varphi, r \operatorname{cos} \theta \operatorname{cos} \varphi, -r \operatorname{sen} \varphi)$$

donde

$$\begin{aligned} \frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \varphi} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r \operatorname{cos} \theta \operatorname{sen} \varphi & r \operatorname{sen} \theta \operatorname{sen} \varphi & 0 \\ r \operatorname{sen} \theta \operatorname{cos} \varphi & r \operatorname{cos} \theta \operatorname{cos} \varphi & -r \operatorname{sen} \varphi \end{vmatrix} \\ &= (r^2 \operatorname{sen} \theta \operatorname{sen}^2 \varphi, r^2 \operatorname{cos} \theta \operatorname{sen}^2 \varphi, r^2 \operatorname{sen} \varphi \operatorname{cos} \varphi) \end{aligned}$$

e

$$\begin{aligned} \left\| \frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \varphi} \right\|^2 &= r^4 \operatorname{sen}^2 \theta \operatorname{sen}^4 \varphi + r^4 \operatorname{cos}^2 \theta \operatorname{sen}^4 \varphi + \\ &\quad + r^4 \operatorname{sen}^2 \varphi \operatorname{cos}^2 \varphi \\ &= r^4 \operatorname{sen}^4 \varphi + r^4 \operatorname{cos}^2 \varphi \\ &= r^4 \operatorname{sen}^2 \varphi \end{aligned}$$

Logo, a área  $A_\sigma$  da superfície de  $\sigma$  é

$$\begin{aligned} A_\sigma &= \int_0^{2\pi} \int_{\varphi_1}^{\varphi_2} \left\| \frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \varphi} \right\| d\varphi d\theta \\ &= \int_0^{2\pi} \int_{\varphi_1}^{\varphi_2} r^2 \operatorname{sen} \varphi d\varphi d\theta \\ &= - \int_0^{2\pi} r^2 \operatorname{cos} \varphi \Big|_{\varphi_1}^{\varphi_2} d\theta \\ &= - \int_0^{2\pi} r^2 (\operatorname{cos} \varphi_2 - \operatorname{cos} \varphi_1) d\theta \\ &= r^2 (\operatorname{cos} \varphi_1 - \operatorname{cos} \varphi_2) \int_0^{2\pi} d\theta \\ &= 2\pi r^2 \left( \frac{h_2}{r} - \frac{h_1}{r} \right) \\ &= 2\pi r (h_2 - h_1) \end{aligned}$$

que é exatamente igual a área do cilindro que está entre os planos que cortam a esfera. ■

**Exercício 4** Observemos inicialmente que

$$\frac{\partial F}{\partial x} = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3x^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3}$$

$$\frac{\partial F}{\partial y} = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3y^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3}$$

$$\frac{\partial F}{\partial z} = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - 3z^2(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3}$$

Assim, temos que

$$\operatorname{div} F(x, y, z) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 0$$

para  $(x, y, z) \neq (0, 0, 0)$ .

Seja  $\sigma$  a esfera unitária de centro na origem, com parametrização

$$\sigma : \begin{cases} x = \operatorname{sen} \theta \operatorname{sen} \varphi \\ y = \cos \theta \operatorname{sen} \varphi \\ z = \cos \varphi \end{cases}$$

Perceba que  $\sigma$  está dentro da região limitada pelo elipsóide  $4x^2 + 9y^2 + 6z^2 = 36$  e além disto

$$\frac{\partial \sigma}{\partial \theta} = (\cos \theta \operatorname{sen} \varphi, -\operatorname{sen} \theta \operatorname{sen} \varphi, 0)$$

$$\frac{\partial \sigma}{\partial \varphi} = (\operatorname{sen} \theta \cos \varphi, \cos \theta \cos \varphi, -\operatorname{sen} \varphi)$$

donde temos que, a normal  $\mathbf{n}$  exterior a  $\sigma$  é dada por

$$\begin{aligned} \mathbf{n} &= \frac{\partial \sigma}{\partial \theta} \times \frac{\partial \sigma}{\partial \varphi} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta \operatorname{sen} \varphi & -\operatorname{sen} \theta \operatorname{sen} \varphi & 0 \\ \operatorname{sen} \theta \cos \varphi & \cos \theta \cos \varphi & -\operatorname{sen} \varphi \end{vmatrix} \\ &= (\operatorname{sen} \theta \operatorname{sen}^2 \varphi, \cos \theta \operatorname{sen}^2 \varphi, \operatorname{sen} \varphi \cos \varphi) \end{aligned}$$

O fluxo de  $F$ , para fora, através de  $\sigma$  é

$$\begin{aligned} \iint_{\sigma} F \cdot \mathbf{n} dS &= \iint_{\sigma} F(\sigma(\theta, \varphi)) \frac{\mathbf{n}}{\|\mathbf{n}\|} \|\mathbf{n}\| d\theta d\varphi \\ &= \iint_{\sigma} F(\sigma(\theta, \varphi)) \cdot \mathbf{n} d\theta d\varphi \end{aligned}$$

mas,

$$\begin{aligned} F(\sigma(\theta, \varphi)) &= F(\operatorname{sen} \theta \operatorname{sen} \varphi, \cos \theta \operatorname{sen} \varphi, \cos \varphi) \\ &= \operatorname{sen} \theta \operatorname{sen} \varphi \vec{i} + \cos \theta \operatorname{sen} \varphi \vec{j} + \\ &\quad + \cos \varphi \vec{k} \end{aligned}$$

e

$$\begin{aligned} F(\sigma(\theta, \varphi)) \cdot \mathbf{n} &= \operatorname{sen}^2 \theta \operatorname{sen}^3 \varphi + \cos^2 \theta \operatorname{sen}^3 \varphi + \\ &\quad + \operatorname{sen} \varphi \cos^2 \varphi \\ &= \operatorname{sen}^3 \varphi + \operatorname{sen} \varphi \cos^2 \varphi \\ &= \operatorname{sen} \varphi \end{aligned}$$

ou seja,

$$\begin{aligned} \iint_{\sigma} F \cdot \mathbf{n} dS &= \iint_{\sigma} \operatorname{sen} \varphi d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^{\pi} \operatorname{sen} \varphi d\varphi d\theta \\ &= 2 \int_0^{2\pi} d\theta \\ &= 4\pi \end{aligned}$$

Assim, considerando  $\phi$  a região externa a  $\sigma$  e interna ao elipsóide  $\psi$ , segue-se que o fluxo de  $F$ , através deste pode se dado por

$$\iint_{\psi} F \cdot \mathbf{n} dS = \iint_{\sigma} F \cdot \mathbf{n} dS + \iint_{\phi} F \cdot \mathbf{n} dS$$

mas,

$$\operatorname{div} F(x, y, z) = 0 \text{ quando } (x, y, z) \in \phi$$

donde, pelo Teorema de Stokes, temos

$$\iint_{\phi} F \cdot \mathbf{n} dS = \iiint_{\phi} \operatorname{div} F dx dy dz = 0$$

Portanto

$$\iint_{\psi} F \cdot \mathbf{n} dS = \iint_{\sigma} F \cdot \mathbf{n} dS = 4\pi$$

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**Exercício 5** O plano que passa pelos pontos  $(1, 0, 0)$ ,  $(0, 1, 0)$  e  $(0, 0, 1)$  possui equação dada por

$$x + y + z = 1$$

e o triângulo de vértices  $(1, 0, 0)$ ,  $(0, 1, 0)$  e  $(0, 0, 1)$  pode ser parametrizado da seguinte forma:

$$\sigma : \begin{cases} x = u \\ y = v \\ z = 1 - u - v \end{cases} \text{ com } \begin{cases} 0 \leq u \leq 1 \\ 0 \leq v \leq 1 - u \end{cases}$$

donde segue-se que

$$\frac{\partial \sigma}{\partial u} = (1, 0, -1)$$

$$\frac{\partial \sigma}{\partial v} = (0, 1, -1)$$

$$\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} = (1, 1, 1)$$

Por outro lado, observe que

$$F(\sigma(u, v)) = F(u, v, 1 - u - v)$$

$$= uv \vec{i} + v(1 - u - v) \vec{j} + u(1 - u - v) \vec{k}$$

$$\operatorname{rot} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= -y \vec{i} - z \vec{j} - x \vec{k}$$

e

$$\operatorname{rot} F(\sigma(u, v)) \cdot \left( \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right) = -v - 1 + u + v - u = -1$$

Usando o teorema de Stokes, temos que

$$\begin{aligned} \oint_C F \cdot dr &= \iint_{\sigma} \operatorname{rot} F \cdot dS \\ &= \iint_{\sigma} \operatorname{rot} F(\sigma(u, v)) \cdot \left( \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \right) dudv \\ &= - \int_0^1 \int_0^{1-u} dvdu \\ &= \int_0^1 (u - 1) du \\ &= \frac{1}{2} u^2 - u \Big|_0^1 \\ &= -\frac{1}{2} \end{aligned}$$

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